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Closed string exchanges on C^2/Z_2 in a background B-field

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Abstract

In an earlier work it was shown that the IR singularities arising in the nonplanar one loop two point function of a noncommutative $\mathcal{N}=2$ gauge theory can be reproduced exactly from the massless closed string exchanges. The noncommutative gauge theory is realised on a fractional D_3 brane localised at the fixed point of the C^2/Z_2 orbifold. In this paper we identify the contributions from each of the closed string modes. The sum of these adds upto the nonplanar two-point function.

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1 Introduction

A generic feature of noncommutative field theories is the coupling of the ultraviolet and the infrared sectors. These theories can be realised as low energy limits of open strings in constant background B-field, in the Seiberg-Witten limit [1]. The loop integrals in the nonplanar diagrams of the noncommutative theories are regulated in the ultraviolet but are divergent when the external momentum goes to zero. This has a natural interpretation in terms of open closed string channel duality where the UV region of the open string channel can be mapped to the IR of the closed string.

As a consequence of the background B-field, the open and closed strings couple to different metrics on the brane and the bulk respectively. We recall that in the presence of a constant background B-field that is nonzero only for the directions (i, j) that are along the brane, the open string modes couple to the metric G. This is related to the metric g that couples to the closed string modes by [1]

$$
\begin{aligned}
G^{MN} &= \left(\frac{1}{g + 2\pi\alpha' B} g \frac{1}{g - 2\pi\alpha' B}\right)^{MN} \\
G_{MN} &= g_{MN} - (2\pi\alpha^{'})^2 (Bg^{-1}B)_{MN}
\end{aligned}
\tag{1}
$$

Noncommutative field theory arises in the following Seiberg-Witten limit,

$$
\alpha^{'} \sim \epsilon^{1/2} \to 0 \;\; ; \;\; g_{ij} \sim \epsilon \to 0 \tag{2}
$$

A framework in which the phenomenon discussed in the opening paragraph can be analysed is the the world-sheet open closed string duality in the presence of a background B-field. It is well known that an open string one loop amplitude can be seen as tree-level exchanges of closed string modes. The circulating modes in the loop degenerates to the massless ones in the open string channel $(t \to \infty)$ and to the closed string massless modes for $(t \to 0)$. Where t is the modulus of the one loop cylinder diagram . Again the UV region of the open string modes correspond to the IR due to the

[1]We will use capital letters $(M, N, ...)$ to denote general spacetime indices and small letters $(i, j, ...)$ for coordinates along the D-brane.

massless closed string states. The correspondence here is between the full tower of open string modes and the massless closed string modes. It would be interesting to analyse situations in which the correspondence is exact between massless modes on either side. The problem outlined earlier in the context of noncommutative gauge theory is analogous to this situation. We have studied this along these lines in the context of the bosonic string theory in [2]. Though we do not expect an exact correspondence in this bosonic model, the analysis has led to various fruitful insights. An exact matching was shown for the $\mathcal{N} = 2$ gauge theory that is realised on a fractional D_3 brane localised at the fixed point of C^2/Z_2 orbifold and the bulk closed string theory [3]. The hypermultiplets in this gauge theory are projected out and the theory is non-conformal. It was further noticed that the role played by the B-field is essentially that of a regulator. Thus leading us to conclude that the UV/IR singularities in noncommutative gauge theory can be seen as IR of the massless closed string exchanges whenever the correspondence between the massless states hold for the commutative models perturbatively. The fact that this is true for the ordinary $\mathcal{N} = 2$ theory was observed in [4]. A class of such models were later analysed in [5, 6, 7, 8]. Also see [10] for reviews.

In this paper, we study the massless closed string exchanges on C^2/Z_2 in a background B-field. We identify the contribution from each of the massless modes. The procedure followed is same as that in [2]. We first derive the couplings for the gauge field with the massless closed string modes from the DBI and the Chern-Simons action. We then compute the nonplanar two point function with two gauge field insertions which then adds upto the one-loop amplitude computed from string theory.

This paper is organised as follows. In Section 2 we review the nonplanar one-loop open string two point amplitude in the closed string channel. In Section 3 we study the closed string exchanges. We first study the massless closed string exchanges for flat space background in Section 3.1. This amplitude vanishes as we can also see from the one loop string computation. Massless exchanges on C^2/Z_2 is the studied in Section 3.2. We end with conclusions in Section 4.

2 Review of one loop amplitude

In this section we review the results of the one loop open string amplitude on the C^2/Z_2 orbifold with two gauge field insertions. Here we give a brief outline of the computation, the details of which may be found in [3].

$$\begin{aligned} A(p,-p) &= iV_4 \det(g+2\pi\alpha' B)\int_0^\infty \frac{dt}{4t}(8\pi^2\alpha' t)^{-2} \times \\ &\times \sum_{(\alpha,\beta,g_i)} Z[{}^\alpha_\beta]_{g_i} \int_0^{2\pi t} dy \int_0^{2\pi t} dy' \left\langle V(p,x,y)V(-p,x',y')\right\rangle_{(\alpha,\beta)} \end{aligned} \quad (3)$$

The factor of $\det(g+2\pi\alpha' B)$ comes from the trace over the world sheet bosonic zero modes. The sum over $(\alpha,\beta)=(0,1/2)$ corresponds to spin structures $(\alpha,\beta)=(0,1/2)$ corresponding to the NS-R sectors and the GSO projection, whereas the sum over g_i projects onto states invariant under the the orbifold action. The elements $Z_{g_i}[{}^\alpha_\beta]$ are the traces over the states computed in [3]. The vertex operator $V(p,x,y)$ is given by,

$$V(p,x,y) = \frac{g_o}{(2\alpha')^{1/2}}\epsilon_j\left(i\partial_y X^j + 4p.\Psi\Psi^j\right)e^{ip.X}(x,y) \quad (4)$$

For the flat space, it is well known that amplitudes with less that four boson insertions vanish. However, in this model the two point amplitude is nonzero. We will now compute this amplitude in the presence of background B-field. First note that the bosonic correlation function, $\left\langle :\partial_y X^i e^{ip.X} :: \partial_y' X^i e^{-ip.X}:\right\rangle$, does not contribute to the two point amplitude as it is independent of the spin structure. The two point function would involve the sum over the $Z_{g_i}[{}^\alpha_\beta]$ which makes this contribution zero as the vacuum amplitude vanishes. The nonzero part of the amplitude will be obtained from the fermionic part,

$$\begin{aligned} \epsilon_k\epsilon_l\left\langle : p.\Psi\Psi^k e^{ip.X} :: p.\Psi\Psi^l e^{-ip.X}:\right\rangle &= \epsilon_k\epsilon_l p_i p_j\left(G^{il}G^{jk} - G^{ij}G^{kl}\right)\times \\ &\times \mathcal{G}^2[{}^\alpha_\beta](w-w')\left\langle : e^{ip.X} :: e^{-ip.X}:\right\rangle \end{aligned} \quad (5)$$

For the planar two point amplitude, both the vertex operators would be inserted at the same end of the cylinder (i.e. at $w=0+iy$ or $\pi+iy$). In this case, the sum in the two point amplitude reduces to,

$$\begin{aligned}\sum_{(\alpha,\beta,g_i)} Z\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right]_{g_i} \mathcal{G}^2\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right](i\Delta y/2\pi) &= \sum_{(\alpha,\beta)} Z\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right]_{e} \mathcal{G}^2\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right](i\Delta y/2\pi) \\ &+ \sum_{(\alpha,\beta)} Z\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right]_{g} \mathcal{G}^2\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right](i\Delta y/2\pi)\end{aligned}$$

$$\begin{aligned} &= \frac{4\pi^2}{\eta(it)^6\vartheta_1^2(i\Delta y/2\pi, it)} \sum_{(\alpha,\beta)} \vartheta^2(0,it)\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right]\vartheta^2\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right](i\Delta y/2\pi, it) + \\ &+ \frac{16\pi^2}{\vartheta_1^2(i\Delta y/2\pi, it)\vartheta_2^2(0,it)}\left[\vartheta_3^2(i\Delta y/2\pi,it)\vartheta_4^2(0,it) - \vartheta_4^2(i\Delta y/2\pi,it)\vartheta_3^2(0,it)\right]\end{aligned} \tag{6}$$

where, $\Delta y = y - y^{'}$. We have separated the total sum as the sum over the two Z_2 group actions. In writing this out we have used the following identity

$$\eta(it) = \left[\frac{\partial_\nu \vartheta_1(\nu, it)}{-2\pi}\right]^{1/3}_{\nu=0} \tag{7}$$

Now, the first term vanishes due to the following identity

$$\begin{aligned}&\sum_{(\alpha,\beta)} \vartheta\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right](u)\vartheta\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right](v)\vartheta\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right](w)\vartheta\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right](s) = \\ &\quad 2\vartheta\begin{bmatrix}1/2\\1/2\end{bmatrix}(u_1)\vartheta\begin{bmatrix}1/2\\1/2\end{bmatrix}(v_1)\vartheta\begin{bmatrix}1/2\\1/2\end{bmatrix}(w_1)\vartheta\begin{bmatrix}1/2\\1/2\end{bmatrix}(s_1)\end{aligned} \tag{8}$$

where,

$$\begin{aligned} u_1 &= \frac{1}{2}(u+v+w+s) \quad v_1 = \frac{1}{2}(u+v-w-s) \\ w_1 &= \frac{1}{2}(u-v+w-s) \quad s_1 = \frac{1}{2}(u-v-w+s) \end{aligned} \tag{9}$$

and noting that, $\vartheta\begin{bmatrix}1/2\\1/2\end{bmatrix}(0,it) = 0$, in the same way as the flat case that makes amplitudes with two vertex insertions vanish. The second term is a constant also due to,

$$\vartheta_4^2(z,it)\vartheta_3^2(0,it) - \vartheta_3^2(z,it)\vartheta_4^2(0,it) = \vartheta_1^2(z,it)\vartheta_2^2(0,it) \tag{10}$$

For the nonplanar amplitude, which we are ultimately interested in, we need to put the two vertices at the two ends of the cylinder such that, $w = \pi + iy$ and $w^{'} = iy^{'}$. It can be seen that the fermionic part of the correlator is constant and independent of t, same as the planar case following from the identity (10). The effect of nonplanarity and the regulation of the two point function due to the background B-field is encoded in the correlation functions for the exponentials. To focus on the closed string exchanges, we study the contribution $t \to 0$ limit of this amplitude. Finally the two point function reduces to,

$$A(p,-p) = iV_4 \det(g + 2\pi\alpha^{'} B) \left(\frac{g_o^2}{8\pi^2\alpha^{'}}\right) \epsilon_k \epsilon_l p_i p_j \left(G^{il}G^{jk} - G^{ij}G^{kl}\right) I(p) \tag{11}$$

where,

$$\begin{aligned} I(p) &= \int ds s^{-1} \exp\left\{-\frac{\alpha^{'}\pi s}{2} p_i g^{ij} p_j\right\} \\ &= 4\pi \int \frac{d^2 k_\perp}{(2\pi)^2} \frac{1}{k_\perp^2 + p_i g^{ij} p_j} \end{aligned} \tag{12}$$

Note that the integral is written in terms of $s = 1/t$ that is again rewritten as an integral over $k_\perp$, the momentum in the directions transverse to the brane for closed strings. The nonzero contribution to the two point amplitude in (3) comes from the $\mathrm{Tr}_{NS}\left[g q^{L_0}\right]$ and $\mathrm{Tr}_{NS}\left[g(-1)^F q^{L_0}\right]$. These correspond to anti-periodic NS-NS and periodic R-R closed strings in the twisted sectors respectively. The fractional D_3-brane is localised at the fixed point of C^2/Z_2. Thus the twisted sector closed string states that couple to it are localised at the fixed point and are free to move in the six directions transverse to the orbifold. This is the origin of the momentum integral (12) in two directions transverse to the D-brane. Before going into the calculation for the closed string exchanges let us break here to discuss the spectrum for the closed strings on this orbifold.

The closed string theory consists of additional twisted sectors apart from the untwisted sectors. The orbifold action an the space-time implies the following boundary conditions on the world-sheet bosons and fermions,

$$
\begin{aligned}
X^I(\sigma+2\pi,\tau) &= \pm X^I(\sigma,\tau) \\
\psi^I(\sigma+2\pi,\tau) &= \pm\psi^I(\sigma,\tau) \quad I=6,7,8,9
\end{aligned} \tag{13}
$$

For the world sheet fermions, the (+)-sign stands for the NS-sector and the (−)-sign for the R-sector. For the other directions the boundary conditions on the world-sheet fields are as usual. We will first list the fields in the untwisted sector. In the NS-NS sector the massless states invariant under the orbifold projection are,

$$
\psi^I_{-1/2}\tilde{\psi}^J_{-1/2}\,|0,k\rangle \tag{14}
$$

where, $I,J=\{2,3,4,5\}$ or $I,J=\{6,7,8,9\}$. The first set of oscillators give the graviton, antisymmetric 2-form field, and the dilaton. The second set gives sixteen scalars.

The orbifold action on the spinor of $SO(8)$ is given by,

$$
|s_1,s_2,s_3,s_4\rangle \rightarrow e^{i\pi(s_3+s_4)}\,|s_1,s_2,s_3,s_4\rangle \tag{15}
$$

The Z_2 invariant R-R state is formed by taking both the left the right states to be either even or odd under Z_2 projection corresponding to $s_3+s_4=0$ or $s_3+s_4=\pm1$ respectively. GSO projection, restricting to both the left and right states to be of the same chirality gives thirty two states. These states correspond to four 2-form fields and eight scalars.

Let us now turn to the twisted sectors. For the twisted sectors the ground state energy for both the NS and the R sectors vanish. In the NS sector the massless modes come from ψ^I_0, $I=6,7,8,9$ oscillators which form a spinor representation of $SO(4)$. With the GSO and the orbifold projections, the closed string spectrum is given by, $2\times2=[0]+[2]$. The [0] and the self-dual [2] constitute the four massless scalars in the NS-NS sector. Similarly, in the R sector, the massless modes are given by ψ^I_0 for $I=2,3,4,5$. Thus giving a scalar and a two-form self-dual field in the closed string R-R sector. The twisted states can also be seen as arising from the dimensional reduction of p-form fields on a vanishing 2-cycle. We will see this more elaborately in Section (3.2). We will write down the couplings of these twisted and untwisted

fields from the DBI and the Chern-Simons action and then compute the contribution from each of these modes, the sum of which would reproduce (11).

3 Closed string exchanges

After reviewing the analysis of the nonplanar two point amplitude, let us now proceed to study in detail the massless closed string exchanges. In this section we will calculate the contribution to the nonplanar two point function with two gauge fields on the brane with massless closed string exchanges coming from the NS-NS and the R-R sectors. As in [2], we will calculate these in three different limits of the closed string metric.

1. In this case the background B-field is assumed to be small and the closed string metric, $g = \eta$. The amplitude will be analysed to $\mathcal{O}(B^2)$.
2. The Seiberg Witten limit when $g = \epsilon\eta$ with the amplitude expanded to $\mathcal{O}(\epsilon^2/(2\pi\alpha^{'})^2)$.
3. The case when the open string metric on the brane, $G = \eta$ so that $g = -(2\pi\alpha^{'})^2 B^2 + \mathcal{O}(\alpha^{'4})$ and the amplitude will be expanded to $\mathcal{O}((2\pi\alpha^{'})^2)$

3.1 Type IIB on flat space

We will start by considering the flat 10D case and consider massless closed string exchanges for a D_3 brane. The two point function will be shown to vanish as expected from the analysis in the previous section. See eqn(6). The results here will be necessary for the later part of this section when we study the exchanges on C^2/Z_2 orbifold. These will precisely be the contributions from the untwisted states upto an overall constant. To begin we write down the supergravity action for the type IIB theory in the Einstein frame [2]

[2] $(A_p \wedge B_q)_{i_1 \dots i_p j_1 \dots j_q} = \frac{(p+q)!}{p!q!} A_{[i_1 \dots i_p} B_{j_1 \dots j_q]}$. $A_p = \frac{1}{p!}\omega_{i_1 \dots i_p} dx^{i_1} \wedge \dots \wedge dx^{i_p}$ and $*A_p = \frac{1}{(d-p)!}\omega_{i_1 \dots d_p} \epsilon^{i_1 \dots d_p}_{j_1 \dots j_{d-p}} dx^{j_1} \wedge \dots \wedge dx^{j_{d-p}}$

$$
\begin{aligned}
S_{IIB} &= \frac{1}{2\kappa^2}\left[\int d^{10}x\sqrt{-g}R - \frac{1}{2}\int \left[d\phi\wedge *d\phi + e^{-\phi}H_3\wedge *H_3\right]\right] \\
&- \frac{1}{4\kappa_{10}^2}\left[\int e^{2\phi}F_1\wedge *F_1 + e^{\phi}F_3\wedge *F_3 + \frac{1}{2}F_5\wedge *F_5\right] + \ldots
\end{aligned} \tag{16}
$$

where $\kappa^2 = \kappa_{10}^2 e^{-2\phi_0}$, and

$$
H_3 = db \quad F_1 = dC_0 \quad F_3 = dC_2 \quad F_5 = dC_4 \tag{17}
$$

Where b is the two form antisymmetric NS-NS field [3]. We have omitted the other terms in the action (16) as we are only interested in the propagators for the closed string modes that will be needed to compute the two point amplitude in the later part of this section. We first write down the propagators for the NS-NS modes that have been worked out in [2] in the context of bosonic string theory.

For the dilaton we have,

$$
\langle\phi\phi\rangle = -2i\kappa^2\frac{1}{k_\perp^2 + g^{ij}k_{\parallel i}k_{\parallel j}} \tag{18}
$$

and for the propagating b field,

$$
\langle b_{IJ}b_{I'J'}\rangle = -\frac{2i\kappa^2}{(2\pi\alpha')^2}\frac{g_{I[J'}g_{I']J}}{k_\perp^2 + g^{ij}k_{\parallel i}k_{\parallel j}} \tag{19}
$$

Note that the factor of $1/(2\pi\alpha')^2$ in the b-field propagator has been included as the sigma model in the noncommutative description is defined with $(2\pi\alpha')B$ coupling. Similarly for the graviton,

$$
\langle h_{IJ}h_{I'J'}\rangle = -2i\kappa^2\frac{[\eta_{I\{J'}\eta_{I'\}J} - 2/(D-2)\eta_{IJ}\eta_{I'J'}]}{k_\perp^2 + k_\parallel^2} \tag{20}
$$

[3]We will denote the constant part of the NS-NS two form field as B and the fluctuation about this as b. The same field on the brane will be identified as the field strength of the $U(1)$ gauge field

The R-R modes that will be relevant for our discussions are the zero-form, C_0 and the two-form, C_2. For the R-R modes, the propagators are same as that of the NS-NS modes upto normalisations,

$$\begin{aligned}
\langle C_0 C_0 \rangle &= \frac{\kappa_{10}^2}{\kappa^2} \langle \phi\phi \rangle \\
\langle C_{2IJ} C_{2I'J'} \rangle &= \frac{\kappa_{10}^2}{\kappa^2} (2\pi\alpha^{'})^2 \langle b_{IJ} b_{I'J'} \rangle
\end{aligned} \tag{21}$$

In the following analysis we shall restrict the propagators to the values for the closed string metric g^{ij} in the various limits stated at the begining of this section. We are interested in the correction to the quadratic term in the effective action for the gauge field on the brane. This can be constructed from the vertices and the propagators for the intermediate massless closed string states. The correction for the nonplanar diagram can be written as,

$$A_2(bb) = \int d^{p+1}\xi \int d^{p+1}\xi^{'} b(\xi) b(\xi^{'}) V < \chi(\xi)\chi(\xi^{'}) > V \tag{22}$$

where,

$$< \chi(\xi)\chi(\xi^{'}) >= \int \frac{d^D k}{(2\pi)^D} < \chi(k_\perp, k_\|)\chi(-k_\perp, -k_\|) > e^{-ik_\|(\xi-\xi^{'})} \tag{23}$$

Where $k_\perp$ is the component of momentum of the closed string mode perpendicular to the brane and $k_\|$ parallel to it. We can rewrite eqn(22) in momentum space coordinates as,

$$\begin{aligned}
A_2(bb) &= V_{p+1} \int \frac{d^{p+1}p}{(2\pi)^{p+1}} b(p) b(-p) \int \frac{d^l k_\perp}{(2\pi)^l} V < \chi(k_\perp, -p)\chi(-k_\perp, p) > V \\
&= V_{p+1} \int \frac{d^{p+1}p}{(2\pi)^{p+1}} b(p) b(-p) L_2(p, -p)
\end{aligned} \tag{24}$$

Where $l = D - (p+1)$ is the number of directions transverse to the brane. In the planar two point function, both the vertices are on the same end of the cylinder in the world-sheet computation. In the field theory this corresponds to putting both the vertices at the same position on the D-brane. In other

words, in the expansions of the DBI and Chern-Simons action, we should be looking for $b^2\chi$ vertices on one end and a χ tadpole on the other. In this case, from the above calculation, $k_{\|} = 0$. So the closed string propagator is just $1/k_{\perp}^2$, i.e. the propagator is not modified by the momentum of the gauge field on the brane. This is what we expect, as in the field theory on the brane, the loop integrals are not modified for the planar diagrams. Here we will only concentrate on the nonplanar sector. Finally to compare with the string theory amplitude, we will identify,

$$b_{kl}(p) \equiv \frac{g_0}{\sqrt{2\alpha^{'}}} F_{kl}(p) = \frac{g_0}{\sqrt{2\alpha^{'}}} p_{[k}A_{l]}(p) \tag{25}$$

3.1.1 NS-NS exchange

We now turn to the DBI and the Chern-Simons action of a D_3 brane for calculating the massless closed string couplings to the gauge field

$$S_{DBI} = -\tau_3 \int d^4\xi \sqrt{g + 2\pi\alpha'(B+b)e^{-\frac{\phi}{2}}} \tag{26}$$

g is the closed string metric in the Einstein frame, B is the constant two form background field and b is the fluctuation of the two form field. As mentioned earlier, the b-field on the brane is interpreted as the two form field strength for the $U(1)$ gauge field and in the bulk it is the usual two form potential. The NS-NS field content is same as that of the bosonic theory. To compute the amplitude due to the exchange of these fields we need to set $D \to 10$ in the amplitudes calculated in [2]. We recollect these expressions for the three cases here. For small B expansion we have,

$$\begin{aligned} L_2 &= -i\kappa^2\tau_3^2(2\pi\alpha^{'})^2 \int \frac{d^6k}{(2\pi)^6} \frac{1}{k_{\perp}^2 + p^2} \times \\ &\times \Big[\frac{(2\pi\alpha^{'})^2}{4} B^{kl}B^{k^{'}l^{'}} + \frac{1}{4}\left[1 - \frac{(2\pi\alpha^{'})^2}{2}\mathrm{Tr}(B^2)\right]\left(\eta^{ll^{'}}\eta^{kk^{'}} - \eta^{lk^{'}}\eta^{kl^{'}}\right) \\ &+ \frac{(2\pi\alpha^{'})^2}{2}\left[(B^2)^{ll^{'}}\eta^{kk^{'}} - (B^2)^{lk^{'}}\eta^{kl^{'}}\right] + (kl) \leftrightarrow (k^{'}l^{'})\Big] \end{aligned} \tag{27}$$

The contribution to (27) comes from the graviton, dilaton and the propagating antisymmetric b field in the bulk. For the noncommutative limit, $g = \epsilon\eta$,

$$L_2 = -i\det(2\pi\alpha^{'} B)\kappa^2\tau_3^2 \int \frac{d^6k_\perp}{(2\pi)^6}\frac{1}{k_\perp^2+\epsilon^{-1}p^2}[\mathcal{O}(1)+\mathcal{O}(\epsilon^2)] \tag{28}$$

where,

$$\mathcal{O}(1) = \left[\frac{1}{4}\left(\frac{1}{B}\right)^{kl}\left(\frac{1}{B}\right)^{k^{'}l^{'}} + (kl) \leftrightarrow (k^{'}l^{'})\right] \tag{29}$$

$$\begin{aligned}
\mathcal{O}(\epsilon^2) &= \frac{\epsilon^2}{(2\pi\alpha')^2}\frac{1}{2}\left[\left[\left(\frac{1}{B^3}\right)^{kl} - \frac{1}{4}\mathrm{Tr}\left(\frac{1}{B^2}\right)\left(\frac{1}{B}\right)^{kl}\right]\left(\frac{1}{B}\right)^{k^{'}l^{'}}\right] \\
&+ \frac{\epsilon^2}{(2\pi\alpha')^2}\left[\frac{1}{4}\left(\frac{1}{B^2}\right)^{kk^{'}}\left(\frac{1}{B^2}\right)^{ll^{'}} - \frac{1}{4}\left(\frac{1}{B^2}\right)^{k^{'}l}\left(\frac{1}{B^2}\right)^{kl^{'}}\right] \\
&+ (kl) \leftrightarrow (k^{'}l^{'})
\end{aligned} \tag{30}$$

For the noncommutative limit, $G = \eta$,

$$L_2 = -i\det(2\pi\alpha^{'} B)\kappa^2\tau_3^2 \int \frac{d^6k_\perp}{(2\pi)^6}\frac{1}{k_\perp^2+\tilde{p}^2(2\pi\alpha^{'})^2}[\mathcal{O}(1)+\mathcal{O}(\alpha^{'2})] \tag{31}$$

$$\mathcal{O}(1) = \left[\frac{1}{4}\left(\frac{1}{B}\right)^{kl}\left(\frac{1}{B}\right)^{k^{'}l^{'}} + (kl) \leftrightarrow (k^{'}l^{'})\right] \tag{32}$$

$$\begin{aligned}
\mathcal{O}(\alpha^{'2}) &= (2\pi\alpha^{'})^2\frac{1}{2}\left[\left[B^{kl} - \frac{1}{4}\mathrm{Tr}(B^2)\left(\frac{1}{B}\right)^{kl}\right]\left(\frac{1}{B}\right)^{k^{'}l^{'}}\right] \\
&+ (2\pi\alpha^{'})^2\left[\frac{1}{4}\left(\eta^{ll^{'}}\eta^{kk^{'}} - \eta^{kl^{'}}\eta^{lk^{'}}\right)\right] \\
&+ (kl) \leftrightarrow (k^{'}l^{'})
\end{aligned} \tag{33}$$

For the noncommutative limits we get contributions only from the dilaton and the antisymmetric b field. The graviton does not contribute to the order to which we are working here.

3.1.2 R-R exchange

The R-R couplings will be given by the usual Chern-Simons terms. We will consider here the commutative description of these terms. For a discussion of noncommutative description see [12, 13]

$$S_{CS} = i\mu_3 \int_4 \sum_n C_n \wedge e^{2\pi\alpha'(B+b)} \tag{34}$$

Expanding (34) and picking out the forms proportional to the volume form with one b insertion we get

$$S_{CS} = i\mu_3 \left[(2\pi\alpha')^2 \int_4 C_0 B \wedge b + (2\pi\alpha') \int_4 C_2 \wedge b \right] \tag{35}$$

<u>C_2 *Exchange*</u> :

The coupling of C_2 to b is given by,

$$V_{bC_2} = \frac{i\mu_3}{4}(2\pi\alpha')\epsilon^{ijkl} \tag{36}$$

The two point amplitude can be worked out as in (24). Using the propagator (21), for small B expansion and $g = \eta$, the contribution to the two point amplitude is,

$$\begin{aligned} L_2(bC_2b) &= \frac{i}{2}\kappa_{10}^2\mu_3^2(2\pi\alpha')^2 \int \frac{d^6k_\perp}{(2\pi)^6}\frac{1}{k_\perp^2+p^2}\left[\frac{1}{2}\left(\eta^{kk'}\eta^{ll'} - \eta^{kl'}\eta^{lk'}\right)\right] \\ &+ (kl) \leftrightarrow (k'l') \end{aligned} \tag{37}$$

For the noncommutative cases, we will rewrite the coupling (36) as

$$
\begin{aligned}
V_{bC_2} &= \sqrt{2\pi\alpha' B}\frac{i\mu_3}{32(2\pi\alpha')}\left(\frac{1}{B}\right)^{pq}\left(\frac{1}{B}\right)^{rs}\epsilon_{pqrs}\epsilon^{ijkl} \qquad (38)\\
&= \sqrt{2\pi\alpha' B}\frac{i\mu_3}{4(2\pi\alpha')}\left[\left(\frac{1}{B}\right)^{ik}\left(\frac{1}{B}\right)^{jl} - \left(\frac{1}{B}\right)^{jk}\left(\frac{1}{B}\right)^{il} - \left(\frac{1}{B}\right)^{ij}\left(\frac{1}{B}\right)^{kl}\right]
\end{aligned}
$$

Where we have used the fact that, for an antisymmetric matrix M of rank $2n$, $\sqrt{M} = \frac{(-1)^n}{2^n n!}\epsilon_{\mu_1..\mu_{2n}}M^{\mu_1\mu_2}...M^{\mu_{2n-1}\mu_{2n}}$. We can now calculate the two point function. Note that the dependence of the amplitude on the closed string metric g only comes from the propagator. For $g = \epsilon\eta$ this is given by,

$$
\begin{aligned}
L_2(bC_2b) = &= i\det(2\pi\alpha' B)\kappa_{10}^2\mu_3^2\frac{\epsilon^2}{(2\pi\alpha')^2}\int\frac{d^6k_\perp}{(2\pi)^6}\frac{1}{k_\perp^2+\epsilon^{-1}p^2}\times\\
&\times\ [\left[\frac{1}{2}\left(\frac{1}{B^3}\right)^{kl} - \frac{1}{8}\mathrm{Tr}\left(\frac{1}{B^2}\right)\left(\frac{1}{B}\right)^{kl}\right]\left(\frac{1}{B}\right)^{k'l'}\\
&+\ \frac{1}{4}\left[\left(\frac{1}{B^2}\right)^{kk'}\left(\frac{1}{B^2}\right)^{ll'} - \left(\frac{1}{B^2}\right)^{k'l}\left(\frac{1}{B^2}\right)^{kl'}\right]\\
&+\ (kl)\leftrightarrow(k'l')] \qquad (39)
\end{aligned}
$$

and for $G = \eta$

$$
\begin{aligned}
L_2(bC_2b) &= i\det(2\pi\alpha' B)\kappa_{10}^2\mu_3^2(2\pi\alpha')^2\int\frac{d^6k_\perp}{(2\pi)^6}\frac{1}{k_\perp^2+\tilde{p}^2/(2\pi\alpha')^2}\times\\
&\times\ \left[\left[\frac{1}{2}B^{kl} - \frac{1}{8}\mathrm{Tr}(B^2)\left(\frac{1}{B}\right)^{kl}\right]\left(\frac{1}{B}\right)^{k'l'} + \frac{1}{4}\left(\eta^{ll'}\eta^{kk'} - \eta^{kl'}\eta^{lk'}\right)\right]\\
&+\ (kl)\leftrightarrow(k'l') \qquad (40)
\end{aligned}
$$

$\underline{C_0\ Exchange}$:

Reading from (35) the coupling of C_0 to the gauge field on the brane is given by

$$V_{bC_0} = \frac{i\mu_3}{4}(2\pi\alpha')^2 B_{ij}\epsilon^{ijkl} \tag{41}$$

The noncommutative couplings can be obtained as the C_2 case, and finally contracting the answer with B_{ij}. This gives

$$V_{bC_0} = \sqrt{2\pi\alpha' B}\frac{i\mu_3}{2}\left(\frac{1}{B}\right)^{kl} \tag{42}$$

And for the two point function, we have

$$\begin{aligned} L_2(bC_0b) &= i\kappa_{10}^2\mu_3^2(2\pi\alpha')^4\int\frac{d^6k_\perp}{(2\pi)^6}\frac{1}{k_\perp^2+p^2}\times \qquad (43)\\ &\times [\frac{1}{4}B^{kl}B^{k'l'} - \frac{1}{8}\mathrm{Tr}(B^2)\left[\eta^{kk'}\eta^{ll'} - \eta^{kl'}\eta^{lk'}\right] \\ &+ \frac{1}{2}\left[\eta^{kk'}(B^2)^{ll'} - \eta^{kl'}(B^2)^{lk'}\right] + (kl)\leftrightarrow(k'l')] \quad \text{(for small } B\text{)} \end{aligned}$$

$$\begin{aligned} L_2(bC_0b) &= i\det(2\pi\alpha')\kappa_{10}^2\mu_3^2\int\frac{d^6k_\perp}{(2\pi)^6}\frac{1}{k_\perp^2+\epsilon^{-1}p^2}\times \\ &\times \left[\frac{1}{4}\left(\frac{1}{B}\right)^{kl}\left(\frac{1}{B}\right)^{k'l'} + (kl)\leftrightarrow(k'l')\right] \quad (\text{for } g=\epsilon\eta) \quad (44) \end{aligned}$$

$$\begin{aligned} L_2(bC_0b) &= i\det(2\pi\alpha')\kappa_{10}^2\mu_3^2\int\frac{d^6k_\perp}{(2\pi)^6}\frac{1}{k_\perp^2+\tilde{p}^2/(2\pi\alpha')^2}\times \\ &\times \left[\frac{1}{4}\left(\frac{1}{B}\right)^{kl}\left(\frac{1}{B}\right)^{k'l'} + (kl)\leftrightarrow(k'l')\right] \quad (\text{for } G=\eta) \quad (45) \end{aligned}$$

It can be seen that, with the identification $\kappa_{10}^2\mu_3^2 = \kappa^2\tau_3^2$, the full contribution to the two point function including both the massless NS-NS and R-R exchanges vanishes i.e.,

$$\begin{array}{rcl} (27) & + & (43)+(37) = 0 \\ (28) & + & (44)+(39) = 0 \\ (31) & + & \underbrace{(45)+(40)} = 0 \\ \text{NS-NS} & + & \quad \text{R-R} \end{array} \tag{46}$$

We know from the one loop string calculation that the one loop two point amplitude vanishes (6). In the closed string picture, this cancellation takes place for every mass-level between the NS-NS and R-R states. We will consider similar exchanges for the massless closed strings on the C^2/Z_2 orbifold in the next section.

3.2 Type IIB on C^2/Z_2 orbifold

We now turn to the massless closed string exchanges on the orbifold that we are ultimately interested in. The procedure followed is same as that of the earlier section. We will first write down the supergravity action on the C^2/Z_2 orbifold. We then derive the couplings of the massless closed string modes to the gauge field on a fractional D_3. In this section we shall primarily make use of the fact that the Z_2 orbifold is the singular limit of a smooth ALE space known as Eguchi-Hanson space [9]. The metric for this space is given by,

$$ds^2 = f(r)^{-1}dr^2 + r^2 f(r)\sigma_z^2 + r^2\left[\sigma_x^2 + \sigma_y^2\right] \tag{47}$$

where,

$$\begin{array}{c} f(r) = \left[1 - \left(\dfrac{a}{r}\right)^4\right] \quad \text{and} \quad \sigma_x = -\dfrac{1}{2}\left(\cos\psi d\theta + \sin\theta\sin\psi d\phi\right) \\ \sigma_y = \dfrac{1}{2}\left(\sin\psi d\theta - \sin\theta\cos\psi d\phi\right) \quad \sigma_z = -\dfrac{1}{2}\left(d\psi + \cos\theta d\phi\right) \end{array} \tag{48}$$

There is an apparent singularity at $r = a$ which is removed if one identifies the range of ψ to be $0 \leq \psi \leq 2\pi$. θ and ϕ have the ranges, $0 \leq \theta \leq \pi$ and $0 \leq \phi \leq 2\pi$. The space near $r = a$ is locally $R^2 \times S^2$. This is seen from the change variables to $u^2 = r^2 \left[1 - \left(\frac{a}{r}\right)^4\right]$, so that for $r = a$ or $u = 0$

$$ds^2 \sim \frac{1}{4}du^2 + \frac{1}{4}u^2 \left(d\psi + \cos\theta d\phi\right)^2 + \frac{a^2}{4}\left(d\theta^2 + \sin^2\theta d\phi^2\right) \tag{49}$$

Note that the R^2 shrinks to a point as $u \to 0$. As $r \to \infty$ the constant r hypersurfaces are given by S^3/Z_2. This is due to the fact that the periodicity of ψ here is 2π instead of the usual periodicity 4π that gives S^3. The orbifold singulatity arises as the radius of the compact 2-sphere reduces to zero size. The compact 2-sphere ($\mathcal{C}_1$) has an associated antiself-dual two form, ω_2 that is dual to $\mathcal{C}_1$ and is given by,

$$\begin{aligned} \omega_2 &= \frac{a^2}{2\pi} d\left(\frac{\sigma_z}{r^2}\right) \\ &= \frac{a^2}{2\pi r^3} dr \wedge d\psi + \frac{a^2}{2\pi r^3}\cos\theta dr \wedge d\phi + \frac{a^2}{4\pi r^2}\sin\theta d\theta \wedge d\phi \end{aligned} \tag{50}$$

This two-form, ω_2 satisfies,

$$\omega_2 = - * \omega_2 \qquad \int_{\mathcal{C}_1} \omega_2 = 1 \qquad \int_{C^2/Z_2} *\omega_2 \wedge \omega_2 = \frac{1}{2} \tag{51}$$

Although the cycle $\mathcal{C}_1$ shrinks to zero size a non-zero two-form flux $\hat{B}$ persists. A $(p+2)$-form may be decomposed as, $A_{p+2} = \tilde{A}_p \wedge \omega_2$. Where $\tilde{A}_p$ is a p-form in the transverse six dimensions. This field is twisted and is localised at the orbifold point. For our analysis we also turn on the $(B + b)$ field along the non-orbifolded directions, so that the background is given by

$$\mathcal{B} = \left(\begin{array}{cccc|ccc} 0 & 1 & 2 & 3 & \ldots & 8 & 9 \\ & & 2\pi\alpha'(B+b) & & & & \\ \hline & & & & & & \hat{B} \end{array}\right) \tag{52}$$

With the above observations and using equations (51), we can now write down the supergravity action on the orbifold for the twisted fields,

$$S_{orb} = -\frac{1}{8\kappa^2}\int_6 d\tilde{b} \wedge *d\tilde{b} - \frac{1}{8\kappa_{10}^2}\int_6 \left[d\tilde{C}_0 \wedge *d\tilde{C}_0 + d\tilde{C}_2 \wedge *d\tilde{C}_2\right] \tag{53}$$

Where $\tilde{b}$ is the twisted NS-NS scalar that arises from the dimensional reduction of the $\hat{B}$ so that $\hat{B} = \hat{b}\omega_2$ and,

$$\hat{b} = 4\pi^2\alpha' \left(\frac{1}{2} + \frac{\tilde{b}}{4\pi^2\alpha'}\right) \tag{54}$$

$\tilde{b}$ is the fluctuating part of $\hat{b}$. Similarly the scalar, $\tilde{C}_0$ and the two-form field $\tilde{C}_2$ arises from the dimensional reduction of the R-R fields C_2 and C_4 respectively. The propagators for these twisted fields can be easily read off from (53),

$$\left\langle \tilde{b}\tilde{b} \right\rangle = -4i\kappa^2 \frac{1}{k_\perp^2 + g^{ij}k_{\|i}k_{\|j}} \tag{55}$$

$$\left\langle \tilde{C}_0\tilde{C}_0 \right\rangle - 4i\kappa_{10}^2 \frac{1}{k_\perp^2 + g^{ij}k_{\|i}k_{\|j}} \tag{56}$$

$$\left\langle \tilde{C}_2\tilde{C}_2 \right\rangle = -4i\kappa_{10}^2 \frac{g_{I[J'}g_{I']J}}{k_\perp^2 + g^{ij}k_{\|i}k_{\|j}} \tag{57}$$

3.2.1 NS-NS exchange

We will now derive the couplings of the gauge field to the twisted NS-NS scalar $\tilde{b}$ that arises from the dimensional reduction of the two form field $\hat{B}$. In the picture outlined in the begining of Section (3.2), we can view a fractional D_p brane as D_{p+2} brane wrapped on the shrinking cycle $\mathcal{C}_1$. The Born-Infeld action for a D_{p+2} is

$$S_{p+2} \;=\; -\tau_{p+2}\int d^{p+3}\xi e^{\frac{p-1}{4}\phi}\sqrt{g+\mathcal{B}e^{-\frac{\phi}{2}}} \tag{58}$$

where $\mathcal{B}$ is given by (52). We can rewrite (58) as,

$$\begin{aligned} S_p &= -\tau_{p+2}\int d^{p+1}\xi e^{\frac{p-3}{4}\phi}\sqrt{g+2\pi\alpha'(B+b)e^{-\frac{\phi}{2}}}\int d^2\xi_{\text{int}}\sqrt{\hat{B}} \\ &= -\tau_p\int d^{p+1}\xi e^{\frac{p-3}{4}\phi}\sqrt{g+2\pi\alpha'(B+b)e^{-\frac{\phi}{2}}}\left(\frac{1}{2}+\frac{\tilde{b}}{4\pi^2\alpha'}\right) \end{aligned} \tag{59}$$

where

$$\int d^2\xi_{\text{int}}\sqrt{\hat{B}} = \int_{\mathcal{C}_1}\hat{B} = 4\pi^2\alpha^{'}\left(\frac{1}{2}+\frac{\tilde{b}}{4\pi^2\alpha'}\right) \tag{60}$$

In the second line of (59) we have identified,

$$\tau_p = \tau_{p+2}(4\pi^2\alpha^{'}) \tag{61}$$

Apart from the usual couplings of the untwisted NS-NS modes, the action gives the coupling of the twisted field $\tilde{b}$. For the untwisted states the couplings and the two point functions are the same as those computed in Section (3.1.1) upto an overall constant. Here we will only be concerned with the twisted field $\tilde{b}$. We can write down the coupling of this field to the gauge field b by expanding (58) with various limits of g,

$$V_{b\tilde{b}} = \frac{2\pi\alpha^{'}}{4\pi}\tau_3 B^{kl} \quad \text{(For small } B \text{ and } g=\eta) \tag{62}$$

$$V_{b\tilde{b}} = \frac{\sqrt{2\pi\alpha' B}}{4\pi^2\alpha'}\tau_3\left[\frac{1}{2}\left(\frac{1}{B}\right)^{kl}+\frac{\epsilon^2}{2(2\pi\alpha')^2}\left[\left(\frac{1}{B^3}\right)^{kl}-\frac{1}{4}\left(\frac{1}{B}\right)^{kl}\text{Tr}\left(\frac{1}{B^2}\right)\right]\right] \quad \text{(For } g=\epsilon\eta) \tag{63}$$

$$V_{b\tilde{b}} = \frac{\sqrt{2\pi\alpha' B}}{4\pi^2\alpha'}\tau_3 \left[\frac{1}{2}\left(\frac{1}{B}\right)^{kl} + \frac{(2\pi\alpha^{'})^2}{2}\left[B^{kl} - \frac{1}{4}\left(\frac{1}{B}\right)^{kl}\mathrm{Tr}\left(B^2\right)\right]\right] \quad (\text{For } G=\eta) \tag{64}$$

The two point amplitudes with the couplings defined above and the propagator (55) are,
For small B,

$$L_2(b\tilde{b}b) = -\frac{i}{4\pi^2}\kappa^2\tau_3^2(2\pi\alpha^{'})^2\int\frac{d^2k_\perp}{(2\pi)^2}\frac{1}{k_\perp^2+p^2}\left[\frac{1}{2}B^{kl}B^{k^{'}l^{'}} + (kl)\leftrightarrow(k^{'}l^{'})\right] \tag{65}$$

For $g = \epsilon\eta$,

$$\begin{aligned} L_2(b\tilde{b}b) &= -\frac{i}{4\pi^2(2\pi\alpha')^2}\det(2\pi\alpha^{'}B)\kappa^2\tau_3^2\int\frac{d^2k_\perp}{(2\pi)^2}\frac{1}{k_\perp^2+\epsilon^{-1}p^2}\times \\ &\times \left[\mathcal{O}(1)+\mathcal{O}(\epsilon^2)\right] \end{aligned} \tag{66}$$

where,

$$\mathcal{O}(1) = \left[\frac{1}{2}\left(\frac{1}{B}\right)^{kl}\left(\frac{1}{B}\right)^{k^{'}l^{'}} + (kl)\leftrightarrow(k^{'}l^{'})\right] \tag{67}$$

$$\begin{aligned} \mathcal{O}(\epsilon^2) &= \frac{\epsilon^2}{(2\pi\alpha')^2}\left[\left(\frac{1}{B^3}\right)^{kl} - \frac{1}{4}\mathrm{Tr}\left(\frac{1}{B^2}\right)\left(\frac{1}{B}\right)^{kl}\right]\left(\frac{1}{B}\right)^{k^{'}l^{'}} \\ &+ (kl)\leftrightarrow(k^{'}l^{'}) \end{aligned} \tag{68}$$

For $G = \eta$,

$$\begin{aligned} L_2(b\tilde{b}b) &= -\frac{i}{4\pi^2(2\pi\alpha')^2}\det(2\pi\alpha^{'}B)\kappa^2\tau_3^2\int\frac{d^2k_\perp}{(2\pi)^2}\frac{1}{k_\perp^2+\tilde{p}^2/(2\pi\alpha')^2}\times \\ &\times \left[\mathcal{O}(1)+\mathcal{O}(\alpha^{'2})\right] \end{aligned} \tag{69}$$

$$\mathcal{O}(1) = \left[\frac{1}{2} \left(\frac{1}{B} \right)^{kl} \left(\frac{1}{B} \right)^{k^{'}l^{'}} + (kl) \leftrightarrow (k^{'}l^{'}) \right] \tag{70}$$

$$\begin{aligned} \mathcal{O}(\alpha^{'2}) &= (2\pi\alpha^{'})^2 \left[B^{kl} - \frac{1}{4}\mathrm{Tr}(B^2) \left(\frac{1}{B} \right)^{kl} \right] \left(\frac{1}{B} \right)^{k^{'}l^{'}} \\ &+ (kl) \leftrightarrow (k^{'}l^{'}) \end{aligned} \tag{71}$$

3.2.2 R-R exchange

Similar to the derivation in Section (3.2.1), we now expand the Chern-Simons action in terms of the twisted and the untwisted R-R fields. Keeping in mind the background two-form field (52) and the relations (51). We start with the action for a D_5 brane wrapping a two cycle $\mathcal{C}_1$.

$$\begin{aligned} S_{CS} &= i\mu_5 \int_6 \sum_n C_n \wedge e^{\mathcal{B}} \\ &= i\mu_5 \frac{1}{2}(4\pi^2\alpha^{'}) \left[(2\pi\alpha^{'})^2 \int_4 C_0 B \wedge b + (2\pi\alpha^{'}) \int_4 C_2 \wedge b \right] \\ &+ i\mu_5 \left[(2\pi\alpha^{'})^2 \int_4 \tilde{C}_0 B \wedge b + (2\pi\alpha^{'}) \int_4 \tilde{C}_2 \wedge b \right] \end{aligned} \tag{72}$$

Identifying $\mu_3 = \mu_5(4\pi^2\alpha^{'})$,

$$\begin{aligned} S_{CS} &= i\mu_3 \frac{1}{2} \left[(2\pi\alpha^{'})^2 \int_4 C_0 B \wedge b + (2\pi\alpha^{'}) \int_4 C_2 \wedge b \right] \\ &+ i\mu_3 \frac{1}{4\pi^2\alpha^{'}} \left[(2\pi\alpha^{'})^2 \int_4 \tilde{C}_0 B \wedge b + (2\pi\alpha^{'}) \int_4 \tilde{C}_2 \wedge b \right] \end{aligned} \tag{73}$$

Note that the twisted and untwisted R-R couplings are same as those computed in Section (3.1.2) except for the change in the overall normalisations.

The R-R exchanges for the twisted states thus have the same tensor structures as those in Section (3.1.2). Incorporating these changes the two point function with twisted R-R exchanges can be written as follows,

$\underline{\tilde{C}_2 \ exchange}$:

For small B,

$$
\begin{aligned}
L_2(b\tilde{C}_2 b) &= \frac{i}{4\pi^2}\kappa_{10}^2\mu_3^2 \int \frac{d^2k_\perp}{(2\pi)^2}\frac{1}{k_\perp^2+p^2}\left[\frac{1}{2}\left(\eta^{kk'}\eta^{ll'}-\eta^{kl'}\eta^{lk'}\right)\right] \\
&+ (kl) \leftrightarrow (k'l')
\end{aligned} \tag{74}
$$

For $g = \epsilon\eta$,

$$
\begin{aligned}
L_2(b\tilde{C}_2 b) = &= \frac{i}{4\pi^2}\det(2\pi\alpha' B)\kappa_{10}^2\mu_3^2 \int \frac{d^2k_\perp}{(2\pi)^2}\frac{1}{k_\perp^2+\epsilon^{-1}p^2} \times \\
&\times \frac{\epsilon^2}{(2\pi\alpha')^4}[\left[\left(\frac{1}{B^3}\right)^{lk} - \frac{1}{4}\mathrm{Tr}\left(\frac{1}{B^2}\right)\left(\frac{1}{B}\right)^{lk}\right]\left(\frac{1}{B}\right)^{l'k'} \\
&+ \frac{1}{2}\left[\left(\frac{1}{B^2}\right)^{kk'}\left(\frac{1}{B^2}\right)^{ll'} - \left(\frac{1}{B^2}\right)^{k'l}\left(\frac{1}{B^2}\right)^{kl'}\right] \\
&+ (kl) \leftrightarrow (k'l')]
\end{aligned} \tag{75}
$$

For $G = \eta$

$$
\begin{aligned}
L_2(b\tilde{C}_2 b) &= \frac{i}{4\pi^2}\det(2\pi\alpha' B)\kappa_{10}^2\mu_3^2 \int \frac{d^2k_\perp}{(2\pi)^2}\frac{1}{k_\perp^2+\tilde{p}^2/(2\pi\alpha')^2} \times \\
&\times \left[\left[B^{kl} - \frac{1}{4}\mathrm{Tr}(B^2)\left(\frac{1}{B}\right)^{kl}\right]\left(\frac{1}{B}\right)^{k'l'} + \frac{1}{2}\left(\eta^{ll'}\eta^{kk'} - \eta^{kl'}\eta^{lk'}\right)\right] \\
&+ (kl) \leftrightarrow (k'l')
\end{aligned} \tag{76}
$$

$\underline{\tilde{C}_0 \text{ } Exchange}$:

$$
\begin{aligned}
L_2(b\tilde{C}_0 b) &= \frac{i}{4\pi^2}\kappa_{10}^2\mu_3^2(2\pi\alpha^{'})^2\int\frac{d^2k_\perp}{(2\pi)^2}\frac{1}{k_\perp^2+p^2}\times \\
&\times \;\; [\frac{1}{2}B^{kl}B^{k^{'}l^{'}} - \frac{1}{4}\mathrm{Tr}(B^2)\left(\eta^{kk^{'}}\eta^{ll^{'}} - \eta^{kl^{'}}\eta^{lk^{'}}\right) \\
&+ \;\; \left[\eta^{kk^{'}}(B^2)^{ll^{'}} - \eta^{kl^{'}}(B^2)^{lk^{'}}\right] + (kl)\leftrightarrow(k^{'}l^{'})] \quad (\text{for small } B)
\end{aligned}
\tag{77}
$$

$$
\begin{aligned}
L_2(b\tilde{C}_0 b) &= \frac{i}{4\pi^2(2\pi\alpha^{'})^2}\det(2\pi\alpha^{'})\kappa_{10}^2\mu_3^2\int\frac{d^2k_\perp}{(2\pi)^2}\frac{1}{k_\perp^2+\epsilon^{-1}p^2}\times \\
&\times \;\; \left[\frac{1}{2}\left(\frac{1}{B}\right)^{kl}\left(\frac{1}{B}\right)^{k^{'}l^{'}} + (kl)\leftrightarrow(k^{'}l^{'})\right] \quad (\text{for } g=\epsilon\eta)
\end{aligned}
\tag{78}
$$

$$
\begin{aligned}
L_2(b\tilde{C}_0 b) &= \frac{i}{4\pi^2(2\pi\alpha^{'})^2}\det(2\pi\alpha^{'})\kappa_{10}^2\mu_3^2\int\frac{d^2k_\perp}{(2\pi)^2}\frac{1}{k_\perp^2+\tilde{p}^2/(2\pi\alpha^{'})^2}\times \\
&\times \;\; \left[\frac{1}{2}\left(\frac{1}{B}\right)^{kl}\left(\frac{1}{B}\right)^{k^{'}l^{'}} + (kl)\leftrightarrow(k^{'}l^{'})\right] \quad (\text{for } G=\eta)
\end{aligned}
\tag{79}
$$

We have seen that the untwisted exchanges for both the NS-NS and R-R sectors are the same as those computed in section (3.1) modulo an overall normalisation. The sum of these thus vanishes just like the flat case, eqns(46). This is also what we get from the one loop computation. See eqn(6). The twisted states however sum up to finite results. We write these contributions below with the identification $\kappa_{10}\mu_3 = \kappa\tau_3$,

$$
\begin{aligned}
L_2 &= (65)+(74)+(77) \quad (\text{for small } B) \\
&= \frac{i}{4\pi^2}\kappa_{10}^2\mu_3^2\int\frac{d^2k_\perp}{(2\pi)^2}\frac{1}{k_\perp^2+p^2}\times \\
&\times \;\; [\frac{1}{2}\left[1-(2\pi\alpha^{'})^2\frac{1}{2}\mathrm{Tr}(B^2)\right]\left(\eta^{kk^{'}}\eta^{ll^{'}} - \eta^{kl^{'}}\eta^{lk^{'}}\right) \\
&+ \;\; (2\pi\alpha^{'})^2\left[\eta^{kk^{'}}(B^2)^{ll^{'}} - \eta^{kl^{'}}(B^2)^{lk^{'}}\right] + (kl)\leftrightarrow(k^{'}l^{'})]
\end{aligned}
\tag{80}
$$

$$
\begin{aligned}
L_2 &= (66)+(75)+(78) \quad (\text{for } g = \epsilon\eta) \\
&= \frac{i}{4\pi^2}\det(2\pi\alpha^{'}B)\kappa_{10}^2\mu_3^2 \int \frac{d^2k_\perp}{(2\pi)^2}\frac{1}{k_\perp^2+\epsilon^{-1}p^2} \times \\
&\times \frac{\epsilon^2}{(2\pi\alpha^{'})^4}\frac{1}{2}\left[\left(\frac{1}{B^2}\right)^{kk^{'}}\left(\frac{1}{B^2}\right)^{ll^{'}} - \left(\frac{1}{B^2}\right)^{k^{'}l}\left(\frac{1}{B^2}\right)^{kl^{'}}\right] \\
&+ (kl) \leftrightarrow (k^{'}l^{'})
\end{aligned}
\tag{81}
$$

$$
\begin{aligned}
L_2 &= (69)+(76)+(79) \quad (\text{for } G = \eta) \\
&= \frac{i}{4\pi^2}\det(2\pi\alpha^{'}B)\kappa_{10}^2\mu_3^2 \int \frac{d^2k_\perp}{(2\pi)^2}\frac{1}{k_\perp^2+\tilde{p}^2/(2\pi\alpha^{'})^2} \times \\
&\times \frac{1}{2}\left[\eta^{ll^{'}}\eta^{kk^{'}} - \eta^{kl^{'}}\eta^{lk^{'}}\right] \\
&+ (kl) \leftrightarrow (k^{'}l^{'})
\end{aligned}
\tag{82}
$$

These are the exact expansions of the one loop sting amplitude, eqn(11) when G and $\det(g+2\pi\alpha^{'}B)$ are expanded to respective orders as follows,

$$
\begin{aligned}
g &= \eta \\
G^{ij} &\sim \eta^{ij} + (2\pi\alpha^{'})^2(B^2)^{ij} + \mathcal{O}(B^4) \\
\sqrt{\eta+(2\pi\alpha^{'})B} &\sim \left[1 - \frac{(2\pi\alpha^{'})^2}{4}\mathrm{Tr}(B^2) + \mathcal{O}(B^4)\right]
\end{aligned}
\tag{83}
$$

$$
\begin{aligned}
g &= \epsilon\eta \\
G^{ij} &\sim -\frac{\epsilon}{(2\pi\alpha^{'})^2}\left(\frac{1}{B^2}\right)^{ij} + \mathcal{O}(\epsilon^3) \\
\sqrt{\epsilon\eta+(2\pi\alpha^{'})B} &\sim \sqrt{(2\pi\alpha^{'})B}\left[1 - \frac{\epsilon^2}{4(2\pi\alpha^{'})^2}\mathrm{Tr}\left(\frac{1}{B^2}\right)\right]
\end{aligned}
\tag{84}
$$

$$
\begin{aligned}
G &= \eta \\
g &= -(2\pi\alpha^{'})^2 B^2 + \mathcal{O}(B^4) \\
\sqrt{g + (2\pi\alpha')B} &\sim \sqrt{(2\pi\alpha')B}\left[1 - \frac{(2\pi\alpha^{'})^2}{4}\mathrm{Tr}(B^2)\right]
\end{aligned} \tag{85}
$$

The computation of the two point amplitude in string theory sums up all the B-field dependence in the open string metric G and $\det(g + 2\pi\alpha^{'}B)$. The analysis in this section reproduces these terms to the orders relevant in the expansion about the various limits of the closed string metric g.

4 Conclusion

In this paper we have studied closed string exchanges in the presence of a background B-field. The main aim is to identify the propagating massless closed string modes that contribute to the non-zero two point amplitude. We have already seen from the one loop string computation in [3], that for the noncommutative $\mathcal{N} = 2$ gauge theory on a fractional D_3 brane, the tree-level massless closed string exchanges give the same result as the one loop gauge theory. Thus the IR divergent terms that arise by integrating over high energy modes in the loops in the nonplanar amplitudes can be interpreted as coming from the massless closed string modes. This follows as a consequence of the world-sheet open closed string duality. This correspondence between the lowest lying modes on either channel is possible as a consequence of cancellation of all massive modes contributing to the loop diagram. Here we have reproduced this amplitude by considering massless closed string exchanges from the effective field theory. Specifically for a fractional D_3 brane localised at the fixed point of C^2/Z_2 orbifold, the closed strings that give finite amplitude come from the twisted NS-NS and R-R sectors. The contributions from the untwisted NS-NS and R-R modes cancel amongst themselves. The duality discussed in the paper was already known for the commutative gauge theory [4]. We have seen that the background B-field primarily acts as a regulator for the non-planar diagrams and preserves this duality. On can thus expect to recover the UV/IR divergences of the noncommutative gauge theory from a finite number massless closed string exchanges in models where

the commutative theory has this duality at least perturbatively. These cases have recently been studied. (See [11] for a review and references).

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